Math 564: Real analysis and measure theory
Lecture 14

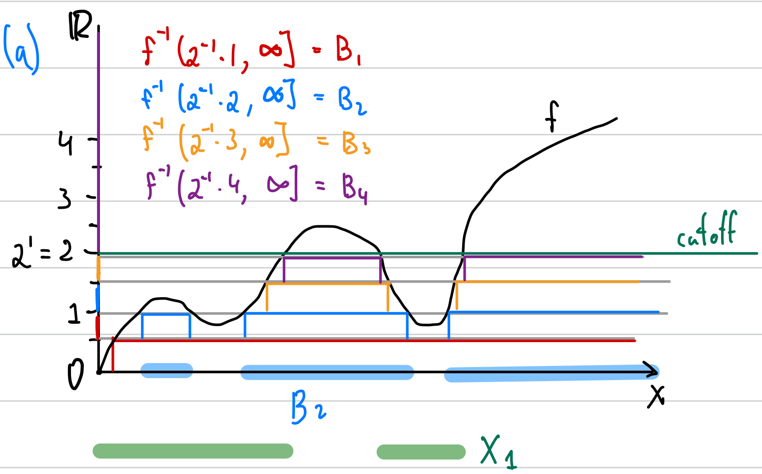
Prop (approximation by simple functions).

(a) For each $f \in L^+(X, \mathcal{B})$, there is an increasing $f_n \nearrow f$ of non-negative simple functions such the convergence is uniform on every set $B \subseteq X$ on which f is bdd, i.e.
$$\| f_n|_B - f|_B \|_u \to 0.$$

(b) For each $f \in L(X, \mathcal{B})$, there is a sequence $f_n \to f$ of simple functions such that $|f_n| \nearrow |f|$ and the convergence is uniform on every set $B \subseteq X$ on which f is bdd.

Proof. (b) follows by apply (a) to f^+ and f^-, thus obtaining non-neg. simple funct. $f_n^+ \nearrow f^+$ and $f_n^- \nearrow f^-$ with uniform convergence statement satisfied, so the functions $f_n := f_n^+ - f_n^-$ are desired.

(a)



To define f_n, we cut off f at $\leq 2^n$ and split $[0, 2^n]$ into pieces of size 2^{-n}, so 2^{2n} pieces. Let $A_k := f^{-1}\left(\left(2^{-n}\cdot k, 2^{-n}(k+1)\right]\right)$ and let $B_k := f^{-1}\left((2^{-n}\cdot k, \infty]\right)$, so $B_0 \supseteq B_1 \supseteq \dots$ and $A_k = B_{k+1} \setminus B_k$. We set
$$f_n := \sum_{k=0}^{2^{2n}-1} (2^{-n}\cdot k) \cdot \mathbb{1}_{A_k} = \sum_{k=1}^{2^{2n}} 2^{-n} \mathbb{1}_{B_k}.$$

Note that $f_n \nearrow f$ since the sets $X_n := f^{-1}([0, 2^n]) \nearrow (X \setminus \{x \in X : f(x) = \infty\})$ so (f_n) converges to f on $\bigcup_{n \in \mathbb{N}} X_n$ and on $X_\infty := \{x \in X : f(x) = \infty\}$, $f_n|_{X_\infty} \equiv 2^n$, so $f_n|_{X_\infty} \nearrow \infty = f|_{X_\infty}$. As for uniform convergence, note that if f is bdd on $B \subseteq X$, then $B \subseteq X_n$ for large enough $n \in \mathbb{N}$ and
$$\| f_n|_{X_n} - f|_{X_n} \|_u \leq 2^{-n} \to 0. \quad \square$$

Def. For $f \in L^+(X, \mathcal{B})$, we define $\int f \, d\mu := \sup \left\{ \int s \, d\mu : 0 \leq s \leq f \text{ simple} \right\}$.

Prop. Let $f, g \in L^+(X, \mathcal{B})$.

(a) $f \le g$ then $\int f \, d\mu \le \int g \, d\mu$.

(b) $\int a \cdot f \, d\mu = a \cdot \int f \, d\mu$ for all $a \ge 0$.

(c) $\int f \, d\mu = 0 \iff f = 0$ a.e.

Proof. (c) $\Rightarrow$. We prove the contrapositive: suppose $X_0 := \{x \in X : f(x) > 0\}$ has positive meas, and show that $\int f \, d\mu > 0$. Indeed, $X_0 = \bigcup_{n \in \mathbb{N}^+} X_n$, where $X_n := \{x \in X : f(x) > \frac{1}{n}\}$, so for some $n \in \mathbb{N}^+$, X_n has positive measure. But $s := \frac{1}{n} \cdot \mathbb{1}_{X_n} \le f$, so $\int f \, d\mu \ge \int s \, d\mu = \frac{1}{n} \cdot \mu(X_n) > 0$. $\square$

Caution. $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$ is not immediate because we don't know whether every simple function $s \le f+g$ splits $s = \hat{f} + \hat{g}$ into a sum of simple functions $\hat{f} \le f$ and $\hat{g} \le g$. To prove this, we need to replace sup with limits of sequences.

Monotone Convergence Theorem (MCT). Let $f_n, f \in L^+(X, \mathcal{B})$. If $f_n \nearrow f$ then $\int f_n \, d\mu \nearrow \int f \, d\mu$.

Proof. We may assume WLOG that $f > 0$ on all of X by restricting to $\{x \in X : f(x) > 0\}$. Because $f_n \le f$, we have $\int f_n \, d\mu \le \int f \, d\mu$ and $\int f_n \, d\mu$ is increasing. We need to show

$$\lim_{n \to \infty} \int f_n \, d\mu \ge \int f \, d\mu.$$

Thus we fix a simple function $0 \le s \le f$ and aim to show $\lim_{n \to \infty} \int f_n \, d\mu \ge \int s \, d\mu$. For this it suffices to fix $\varepsilon > 0$ and show that

$$\lim_{n \to \infty} \int f_n \, d\mu \ge \int (1-\varepsilon) s \, d\mu$$

because $\int (1-\varepsilon) s \, d\mu = (1-\varepsilon) \int s \, d\mu \nearrow 1$ as $\varepsilon \to 0$. Because $(1-\varepsilon) s < f$ for all $x \in X$, we have that $X = \bigcup_{n \in \mathbb{N}}^{\uparrow} X_n$ where $X_n := \{x \in X : f_n(x) > (1-\varepsilon) s\}$. By monotonicity,

monotone convergence for measures

$$\int f_n \, d\mu \ge \int_{X_n} f_n \, d\mu \ge \int_{X_n} (1-\varepsilon) s \, d\mu = \mu_{(1-\varepsilon)s}(X_n) \underset{n \to \infty}{\nearrow} \mu_{(1-\varepsilon)s}(X) = \int (1-\varepsilon) s \, d\mu.$$

$\square$

<u>Corollary (ctbl additivity of integral).</u> The integral on $L^+(X,\mathcal{B})$ is ctbly additive, i.e.

$$\int \sum_{n\in\mathbb{N}} f_n \, d\mu = \sum_{n\in\mathbb{N}} \int f_n \, d\mu.$$

Proof. Let's show first that $\int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu$. Let $f_n \nearrow f$ and $g_n \nearrow g$ be simple functions, so $f_n + g_n \nearrow f+g$. By MCT, we have:

$$\int f+g \, d\mu \overset{MCT}{=} \lim_{n\to\infty} \int (f_n+g_n) \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu + \lim_{n\to\infty} \int g_n \, d\mu \overset{MCT}{=} \int f \, d\mu + \int g \, d\mu.$$

For an infinite sum $\sum_{n\in\mathbb{N}} f_n$, observe that $\sum_{n\le N} f_n \underset{N\to\infty}{\nearrow} \sum_{n\in\mathbb{N}} f_n$ so by MCT again, we have:

$$\sum_{n\in\mathbb{N}} \int f_n \, d\mu = \lim_{N\to\infty} \sum_{n<N} \int f_n \, d\mu = \lim_{N\to\infty} \int \sum_{n<N} f_n \, d\mu = \int \sum_{n\in\mathbb{N}} f_n \, d\mu.$$ □

<u>Corollary (functions defining measures).</u> Each $f \in L^+(X,\mathcal{B})$ defines a measure μ_f on $(X,\mathcal{B})$ by

$$\mu_f(B) := \int \mathbb{1}_B \cdot f \, d\mu =: \int_B f \, d\mu.$$

Proof. Clearly, $\mu_f(\emptyset) = 0$ so let $B = \bigsqcup_{n\in\mathbb{N}} B_n$ with all sets in $\mathcal{B}$. Then $\mathbb{1}_B = \sum_{n\in\mathbb{N}} \mathbb{1}_{B_n}$, so $\mathbb{1}_B \cdot f = \sum_{n\in\mathbb{N}} (\mathbb{1}_{B_n} \cdot f)$, so by ctbl additivity:

$$\mu_f(B) = \int \mathbb{1}_B \cdot f \, d\mu = \sum_{n\in\mathbb{N}} \int \mathbb{1}_{B_n} f \, d\mu = \sum_{n\in\mathbb{N}} \mu_f(B_n).$$ □

<u>Fatou's lemma.</u> Let $\{f_n\} \subseteq L^+(X,\mathcal{B})$. Then $\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n \, d\mu$.

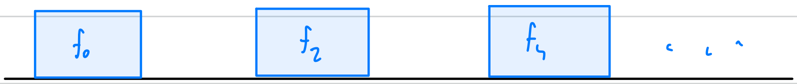
Proof. $\liminf_{n\to\infty} f_n := \lim_{n\to\infty} \left(\inf_{i\ge n} f_i \right)$, so $\inf_{i\ge n} f_i \underset{n\to\infty}{\nearrow} \liminf_{n\to\infty} f_n$, so by MCT:

$$\lim_{n\to\infty} \int \inf_{i\ge n} f_i \, d\mu = \int \liminf_{n\to\infty} f_n \, d\mu.$$

$\int \inf_{i\ge n} f_i \, d\mu \le \int f_m \, d\mu$ for all $m \ge n$, so $\int \inf_{i\ge n} f_i \, d\mu \le \inf_{m\ge n} \int f_m \, d\mu$, hence

$$\liminf_{n\to\infty} f_n \, d\mu \le \lim_{n\to\infty} \inf_{m\ge n} \int f_m \, d\mu = \liminf_{n\to\infty} \int f_n \, d\mu.$$ □

__Example__ (of strict inequality in Fatou). For $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, let $f_n := \mathbb{1}_{[n,n+1)}$, so



$f_n \to 0$ but $\int f_n \, d\lambda = 1$ for all $n \in \mathbb{N}$.